6.390 Introduction to Machine Learning
Recitation Week #10
Issued November 14, 2022

1. Emmy Tea wants to start selling her Fizz Water using a vending machine. Each beverage will cost forty cents and the vending machine will only accept one quarter or one dime at a time. Once forty or more cents are entered into the vending machine, the Fizz Water will be automatically dispensed and the vending machine will return to its initial state. The vending machine will not return change; if you pay more than 40 cents, you lose the extra money.

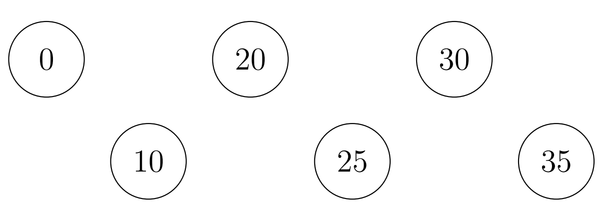
   Emmy wants to model her vending machine using a finite state machine, whose inputs are the coins a user enters and whose output is the digital display of how much money the user has entered so far toward the next beverage. She decides that she will use six states, each corresponding to a monetary amount:

$$\{0, 10, 20, 25, 30, 35\}.$$

   (a) What are the possible input(s) to the state machine?

   (b) What are the possible outputs of the state machine?

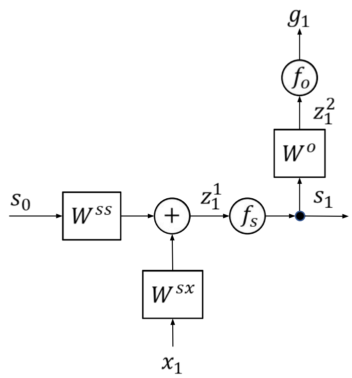
   (c) In the diagram below, draw arrows to indicate all of the possible state transitions, and label those arrows with the input that caused that transition.



2. Consider the following recurrent neural network (RNN):

$$z_t^1 = W^{ss} s_{t-1} + W^{sx} x_t, \qquad s_t = f_s(z_t^1),$$
$$z_t^2 = W^o s_t, \qquad g_t = f_o(z_t^2),$$

where we have set biases to zero. Here, we have a data set $\mathcal{D}_n = \left\{(x^{(i)}, y^{(i)})\right\}_{i=1}^{q}$ consisting of input-output sequence pairs. The i^{th} sequence has length $n^{(i)}$. The output of the RNN is designated by a sequence $g^{(i)} = \text{RNN}(x^{(i)}; W)$, where W is an object which consists of the weight matrices $\{W^{ss}, W^{sx}, W^o\}$. In the figure below is a visualization of one stage of the RNN.



(a) Assume our first RNN, call it RNN-A, has s_t, x_t, g_t all being vectors. Let s_t be of shape 2×1, x_t of shape 3×1 and g_t of shape 4×1. In addition, the activation functions are $f_s(z) = z$ and $f_o(z) = z$.

i. For RNN-A, give dimensions of the following vectors and matrices:

W^{ss}: ____________ W^{sx}: ____________ W^o: ____________

z_t^1: ____________ z_t^2: ____________

ii. Consider a particular RNN-A where all the elements of W^{ss}, W^{sx}, and W^o are 1. Let's look at an input sequence x of length 1. The first (and only) element of this sequence is $x_1 = [1, 0, 0]^\top$. Our initial state is $s_0 = [1, 1]^\top$. Since the input sequence had length 1, the output sequence will also have length 1. What is the output of RNN-A, g_1?

(b) Using the structure of RNN-A, we wish to implement a couple of different functionalities.

We've got a lot of dimensions we're using. We have a sequence x of vectors (and we could have multiple sequences, which we would then distinguish with a superscript). The t^{th} element of our sequence is the vector x_t, which in turn has multiple elements. Here, we'll use bracket notation $x_t[i]$ to denote the i^{th} element of the vector x_t.

i. First, we want to implement weight matrices and initial states such that the output g_t will be:

$$g_t = \begin{bmatrix} x_t[1] \\ 0 \\ 0 \\ \sum_{j=1}^{t} \sum_{i=1}^{3} x_j[i] \end{bmatrix}.$$

That is, the first element of g_t is the first element of x_t, and the last element of g_t is the sum of all of the elements of x over all inputs x_j, for $j = 1, \dots, t$.

Define s_0, W^{ss}, W^{sx}, and W^o necessary to implement the behavior described above.

ii. Now, we want weight matrices and initial state to implement:

$$g_t = \begin{bmatrix} x_t[1] \\ x_t[2] \\ x_t[3] \\ \sum_{j=1}^{t} \sum_{i=1}^{3} x_j[i] \end{bmatrix}.$$

That is, to have the output $g_t[2] = x_t[2]$ and $g_t[3] = x_t[3]$, in addition to the outputs from part (b) i. If we keep the state at size 2×1 and the activation functions as $f_s(z) = z$ and $f_o(z) = z$, is it possible to implement this with the RNN-A structure? Why or why not?

(c) Now consider a modified RNN, call it RNN-B, that does the following:

$$z_t^1 = \begin{bmatrix} s_{t-1} \\ x_t \end{bmatrix}, \qquad s_t = f_s\left(W^{ssx} z_t^1\right),$$
$$z_t^2 = \begin{bmatrix} s_t \\ x_t \end{bmatrix}, \qquad g_t = f_o\left(W^{ox} z_t^2\right),$$

where s_t, x_t, g_t are all vectors. Let s_t be of shape 2×1, x_t of shape 3×1 and g_t of shape 4×1. Then, $\begin{bmatrix} s_{t-1} \\ x_t \end{bmatrix}$ and $\begin{bmatrix} s_t \\ x_t \end{bmatrix}$ are the concatenation of the two column vectors into a new column vector. In addition, the activation functions are $f_s(z) = z$ and $f_o(z) = z$.

i. For RNN-B, give dimensions of the following matrices:

W^{ssx}: __________ W^{ox}: __________ z_t^1: __________ z_t^2: __________

ii. Now, would it be possible for RNN-B to implement the functionality discussed in part (b) ii? If so, define s_0, W^{ssx}, and W^{ox} necessary to implement the behavior.

iii. Instead of using RNN-B, could we change the state space representation s_t and weights in our standard RNN structure (RNN-A) to achieve the capabilities of RNN-B?