Name: ______________________________

Delay lines

7. (10 points) Recall the specification of a standard recurrent neural network (RNN): input x_t of dimension $\ell \times 1$, state s_t of dimension $m \times 1$, and output y_t of dimension $v \times 1$. The weights in the network, then, are

$$W^{sx} : m \times \ell$$
$$W^{ss} : m \times m$$
$$W^{O} : v \times m$$

with activation functions f_1 and f_2. **Throughout this problem, for simplicity, we will treat all offsets as equal to 0.** Finally, the operation of the RNN is described by

$$s_t = f_1\left(W^{sx} x_t + W^{ss} s_{t-1}\right)$$
$$y_t = f_2\left(W^{O} s_t\right) \quad .$$

(a) Consider an RNN defined by $\ell = 1$, $m = 2$, $v = 1$, $f_1 = f_2 =$ the identity function, and

$$W^{sx} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad W^{ss} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad W^{O} = \begin{bmatrix} -3 & -2 \end{bmatrix}$$

Assuming the initial state is all 0, and the input sequence is $[[1], [-1]]$, what is the output sequence?

Solution:

$$s1 = [5, 6]^T$$
$$y1 = -15 - 12 = -27$$
$$s2 = [-5, -6]^T + [5 + 12, 15 + 24]^T = [12, 33]^T$$
$$y2 = -36 - 66 = -102$$

So answer is $[[-27], [-102]]$. Don't worry about transpose.

(b) We can think of the RNN as mapping input sequences to output sequences. Jody thinks that if we remove f_1 and f_2 then the mapping from input sequence to output sequence can be achieved by a hypothesis of the form $Y = WX$. In the case of a length 3 sequence, assuming inputs and outputs are 1-dimensional, $s_0 = [0]$, $X = [x_1, x_2, x_3]^T$, $Y = [y_1, y_2, y_3]^T$, and W is 3×3.

i. Is Jody right? $\surd$ **Yes** ◯ No

ii. If Jody is right, provide a definition for W in Jody's model in terms of W^{sx}, W^{ss}, and W^O of the original RNN that makes them equivalent If Jody is wrong, explain why.

Solution:

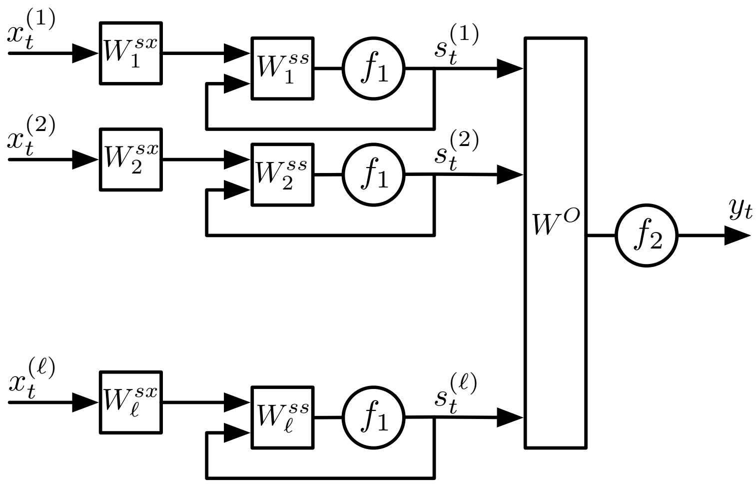
$$W = \begin{bmatrix} W^O W^{sx} & 0 & 0 \\ W^O W^{ss} W^{sx} & W^O W^{sx} & 0 \\ W^O W^{ss} W^{ss} W^{sx} & W^O W^{ss} W^{sx} & W^O W^{sx} \end{bmatrix}$$

(c) Pat thinks a different RNN model would be good. Its operation is defined by

$$s_t^{(i)} = f_1\left(W_i^{sx} x_t^{(i)} + W_i^{ss} s_{t-1}^{(i)}\right)$$
$$y_t = f_2\left(W^O s_t\right) \quad .$$

where the dimension of the state, $m = k \cdot \ell$, so there are k state dimensions for each input dimension, $s^{(i)}$ is the ith group of k dimensions in the state vector, $x^{(i)}$ is the ith entry in the input vector, W_i^{sx} is $k \times 1$ and W_i^{ss} is $k \times k$.

Here is a diagram.



i. Can this model represent the same set of state machines as a regular RNN?

◯ Yes √ **No**

ii. If yes, explain how to convert the weights of a regular RNN into weights for Pat's model.

If no, describe a concrete input/output relationship (for example, the output y_t is the sum of all the inputs $x_t^{(1)}, \ldots, x_t^{(\ell)}$) that **can** be represented by a regular RNN but cannot be represented by Pat's model, for any value of k.

Solution: Output a 1 if and only if $x^{(1)}$ and $x^{(2)}$ were simultaneously non-zero.