Trial separation

Let's look at linear separability and linear classification.

3. (10 points) Linear separability.

   (a) Consider the following $n = 4$ data set with 4-dimensional data points (recall that each column represents one data point):

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} +1 & +1 & -1 & -1 \end{bmatrix}$$

   Is the data linearly separable? If yes, please provide a classifier θ, θ_0 that correctly classifies the data. If no, please explain why not.

   > **Solution:** Yes, the data is separable. The first feature (the top row) of X completely determines Y. One solution is $\theta^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \theta_0 = -.5$.

   For each of the following True/False questions, please provide a brief explanation following your answer.

   (b) If we take any linearly separable data set and *add* a new feature, it is still guaranteed to be linearly separable.

   √ **True** ◯ False

   > **Solution:** Assume that model had d features and it was separable. When adding the new feature, the associated weight θ_{d+1} can be set to 0. That would keep the data separable, as before.

   (c) If we take any linearly separable data set and *remove* a feature, it is still guaranteed to be linearly separable.

   ◯ True √ **False**

   > **Solution:** Assume that the data was separable due to just one feature. The data set provided in part (a) is one such case where, for two of the data points, the first feature is the only differentiation between the positive and negative labels. Now assume that we remove the first feature in that example. The resulting data would no longer be separable (there are data points with identical features but different labels).

(d) If we take any data set that is not linearly separable and *remove* a feature, it is still guaranteed to not be linearly separable.

√ **True** ◯ False

Solution: Removing a feature would be the same as setting the weight associated with the removed feature to be $= 0$ in the original data set. However, since the original data set was not linearly separable, there is no setting of weights (including one with the removed feature's weight set to 0) that correctly classifies all of the data.

(e) If we take any data set that is not linearly separable and remove a *data point*, it is still guaranteed to not be linearly separable.

◯ True √ **False**

Solution: Consider the following datapoints and associated labels:

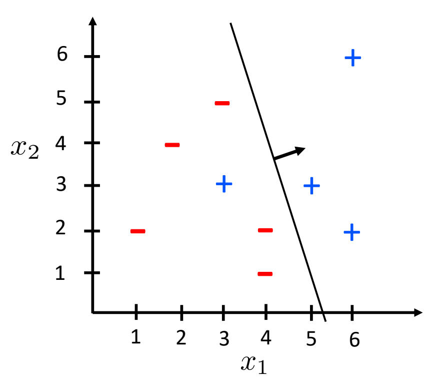
$$\begin{aligned}(1,1) &: +1,\\ (1,-1) &: -1,\\ (-1,-1) &: +1,\\ (-1,1) &: -1\end{aligned}$$

This data set is not linearly separable (remember XOR?). Imagine we remove the datapoint $(1,1)$. Now, the remaining data is linearly separable.

4. (6 points) Consider the data set shown in the box below.

(a) Draw a hyperplane that obtains the smallest training error (i.e., highest accuracy). Be sure to also draw the normal vector.

Solution:



(b) Suppose we remove data points ($x_1 = 3$, $x_2 = 3$) and ($x_1 = 4$, $x_2 = 2$). And let us say that two hypotheses are considered different if there exists a test point (i.e., not necessarily from the data set shown) that they would classify differently.

How many different hypotheses are there that obtain zero training error? Explain your answer.

Solution: There is an infinite number of hypotheses that can obtain zero training error because the dataset is now linearly separable. These correspond to the infinite number of straight lines that can lie between the positively labeled data point $(5, 3)$ and the two negatively labeled data points $(4, 1), (3, 5)$.